

GENERALIZATION OF SCHWARZ-PICK LEMMA TO INVARIANT VOLUME IN A KÄHLER MANIFOLD

BY

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1. Introduction. Let \mathcal{D} be the class of bounded homogeneous star-shaped domains $D \subset \text{space } C^n$ of n complex variables $z = (z^1, \dots, z^n)$. A domain D is homogeneous if any point of D can be transformed into any other by a holomorphic automorphism; D is star-shaped with respect to a point $z_0 \in D$ if $z \in D$ implies that $r(z - z_0) \in D$ for $0 < r \leq 1$. A bounded domain D possesses the Bergman metric, which is invariant under biholomorphic mappings. Let \mathcal{K} be a class of Kähler manifolds Δ such that the components of its Ricci curvature tensor satisfy certain boundedness conditions (see formulas (2.9)). We consider biholomorphic mappings $w = w(z)$ of $D \in \mathcal{D}$ into $\Delta \in \mathcal{K}$, that is, $w = (w^1, \dots, w^n)$, where w is local coordinate on Δ , and $w^j = w^j(z^1, \dots, z^n)$, are holomorphic functions on D with Jacobian determinant

$$J_w(z) \equiv \partial(w)/\partial(z) \neq 0.$$

In §2 we generalize the Ahlfors version of the Schwarz-Pick lemma in C^1 to invariant volume in bounded homogeneous domains $D \in \mathcal{D}$ in C^n . This theorem states that if $w = w(z)$ is a holomorphic mapping of the disk $|z| < 1$ into a Riemann surface W and if the metric $d\sigma = \lambda |dw|$, $\lambda > 0$, of W has a negative curvature ≤ -4 everywhere on W , then

$$\lambda |dw/dz| \leq 1/(1 - |z|^2)$$

for $|z| < 1$ [1]. Theorem 1 gives the invariant form of this generalization (with respect to biholomorphic mappings) and Theorem 2 an inequality which generalizes an inequality obtained by Dinghas when D is the unit hypersphere [4]. The proof of these theorems uses the method of Ahlfors in [1] and depends on properties of certain relative invariants of D , in particular, the fact that the Bergman kernel function of a bounded homogeneous domain is infinite everywhere on the boundary (Lemma 1). In §3 various applications and extensions of the ideas in §2 are given in Theorems 3-6 and corollary. In §4 the results of §2 are applied to a study of the relative invariants of the classical Cartan domains R_j ($j = \text{I}, \dots, \text{IV}$), in particular, the invariant I_j (see (2.4)) is calculated. This procedure leads to a solution of a certain nonhomogeneous partial differential equation formed from the Hessian determinant of a holomorphic function on R_j .

2. Generalization of Schwarz-Pick lemma. 1. Let D be a bounded homogeneous star-shaped domain of class \mathcal{D} . Its Bergman metric is given by

$$(1) \quad ds_D^2 = T_{\alpha\beta} dz^\alpha d\bar{z}^\beta$$

(the summation convention is used), where

$$T_{\alpha\beta} = T_{\alpha\beta}(z, \bar{z}) = \frac{\partial^2 \log K_D(z, \bar{z})}{\partial z^\alpha \partial \bar{z}^\beta},$$

(2)

$$T_D = T_D(z, \bar{z}) = \det(T_{\alpha\beta}),$$

$K_D(z, \bar{z})$ the Bergman kernel function of D [2]. The Bergman metric is a Kähler metric on D which is invariant under biholomorphic mappings of D . The domains D in \mathcal{D} have the additional properties:

(i) The image domains D_r of D under the similarity transformation

$$(3) \quad w = r(z - z_0), \quad 0 < r \leq 1,$$

are such that $D_{r_1} \subset D_{r_2}$ if $r_1 \leq r_2$. Also $D = \bigcup_{j=1}^\infty D_{r_j}$, where r_j , $0 < r_j < 1$, is an increasing sequence with limit 1. These facts follow since D is star-shaped. (Without loss of generality we may take $z_0 = 0$ in (3).)

(ii) The kernel function $K_D(z, \bar{z})$ becomes infinite on the boundary ∂D of D . (This means that the set $\{z: z \in D \text{ and } K_D(z, \bar{z}) < M\}$ is relatively compact on D .) This result is included in Lemma 1.

DEFINITION. A real-valued function $R_D(z, \bar{z})$ on D is a relative invariant of D if under any biholomorphic mapping $w: D \rightarrow D^*$

$$R_{D^*}(w, \bar{w}) |J_w(z)|^2 = R_D(z, \bar{z}).$$

The functions $K_D(z, \bar{z})$ and $T_D(z, \bar{z})$ are relative invariants of D [2] and consequently the function

$$(4) \quad I_D(z, \bar{z}) = K_D(z, \bar{z})/T_D(z, \bar{z})$$

is invariant under biholomorphic mappings:

$$(5) \quad I_D(z, \bar{z}) = I_{D^*}(w, \bar{w}).$$

It is clear that if J_D is another invariant of D , then $J_D = kI_D$ for some constant k . Therefore, an invariant on a homogeneous domain is uniquely determined up to a constant multiple.

LEMMA 1. Any relative invariant $R_D(z, \bar{z})$ of a bounded homogeneous domain D becomes infinite on ∂D .

Proof. Let Γ be the group of holomorphic automorphisms of D . Since the set of elements of Γ is uniformly bounded it forms a normal family. Let a be an

arbitrary point and a_0 a fixed point of D . Since D is homogeneous there is an automorphism $t=t_a(z)$ which maps a_0 into a . Since $R_D(a, \bar{a})$ is a relative invariant of D

$$(6) \quad R_D(a, \bar{a}) = R_D(a_0, \bar{a}_0) |J_{t_a}(a_0)|^{-2}.$$

Let $b \in \partial D$. It follows from a well-known theorem of H. Cartan [3] that

$$(7) \quad \lim_{a_i \rightarrow b} J_{t_{a_i}}(z) = 0.$$

The lemma follows from (6) and (7).

Let \mathcal{K} be the class of Kähler manifolds Δ with metric given by

$$(8) \quad d\sigma_\Delta^2 = g_{\alpha\beta}(w, \bar{w}) dw^\alpha d\bar{w}^\beta,$$

$$g_\Delta = g_\Delta(w, \bar{w}) = \det(g_{\alpha\beta}),$$

where w is a local coordinate of a point on Δ . We also assume

$$(9a) \quad -r_{\alpha\beta} u^\alpha \bar{u}^\beta \geq 0,$$

$$(9b) \quad \det(-r_{\alpha\beta}) \geq g_\Delta,$$

where

$$(9c) \quad r_{\alpha\beta} = -\frac{\partial^2 \log g_\Delta}{\partial w^\alpha \partial \bar{w}^\beta}$$

are the components of the Ricci curvature tensor of the metric (8) [8, 126]. Since $I_D = K_D/T_D$ is constant for a homogeneous domain, it follows that the components of the Ricci curvature tensor of the metric (1) have the form

$$-\frac{\partial^2 \log T_D}{\partial z^\alpha \partial \bar{z}^\beta} = -T_{\alpha\beta}$$

so that (9) is satisfied. Hence \mathcal{D} is a subclass of \mathcal{K} .

2. THEOREM 1. *If a bounded homogeneous domain D of class \mathcal{D} can be mapped biholomorphically by $w=w(z)$ into a Kähler manifold $\Delta \in \mathcal{K}$, then*

$$(10) \quad g_\Delta(w, \bar{w}) |J_w(z)|^2 \leq T_D(z, \bar{z})$$

on D . Equality holds if the mapping is onto and the Kähler metric of Δ equals the Bergman metric of Δ .

Proof. Let $z \in D$. By (i) there exists an $r < 1$, such that $z \in D_r$. Now D_r is a homogeneous domain: $z_1, z_2 \in D_r$ implies $z_1/r, z_2/r \in D$ by the similarity transformation s given by (3) with $z_0=0$; thus there is an automorphism t of D which takes z_1/r into z_2/r and $s^{-1}ts$ is an automorphism of D_r and takes z_1 into z_2 . Let $\zeta = \zeta(z)$ be an automorphism of D_r which takes an arbitrary point z of D_r into 0. From the definition of relative invariant

$$(11) \quad I_{D_r} = \frac{K_{D_r}(0, 0)}{T_{D_r}(0, 0)} = \frac{K_{D_r}(z, \bar{z})}{T_{D_r}(z, \bar{z})} = I_{D_r}(z, \bar{z})$$

so that the invariant $I_{D_r}(z, \bar{z})$ is a constant on D_r , and by (5)

$$(12) \quad I_{D_r} = I_D.$$

Let

$$G_{\alpha\beta} dz^\alpha d\bar{z}^\beta$$

be the hermitian form on D corresponding to the metric (8) on Δ under the inverse mapping $z=z(w)$ of $w: D \rightarrow \Delta$ [5, 79]. Then

$$G_{\mu\nu}(z, \bar{z}) = g_{\alpha\beta}(w, \bar{w}) \frac{\partial w^\alpha}{\partial z^\mu} \frac{\partial \bar{w}^\beta}{\partial \bar{z}^\nu}$$

and

$$(13) \quad G_D(z, \bar{z}) = g_\Delta(w, \bar{w}) |J_w(z)|^2 > 0.$$

Let

$$R_{\alpha\beta}(z, \bar{z}) = \frac{-\partial^2}{\partial z^\alpha \partial \bar{z}^\beta} \log G_D(z, \bar{z}).$$

Then

$$r_{\alpha\beta}(w, \bar{w}) \frac{\partial w^\alpha}{\partial z^\mu} \frac{\partial \bar{w}^\beta}{\partial \bar{z}^\nu} = R_{\mu\nu}(z, \bar{z})$$

and

$$\det(-r_{\alpha\beta}(w, \bar{w})) = \det(-R_{\alpha\beta}(z, \bar{z})) |J_w(z)|^{-2}.$$

Thus from hypothesis (9b)

$$(14) \quad \det(-R_{\alpha\beta}(z, \bar{z})) \geq g_\Delta(w, \bar{w}) |J_w(z)|^2 = G_D(z, \bar{z}).$$

Following the proof in Dinghas and Ahlfors [4, 11] let

$$(15) \quad U = \log \frac{G_D(z, \bar{z})}{T_D(0, 0)}, \quad V = \log \frac{K_{D_r}(z, \bar{z})}{K_D(0, 0)},$$

$$\Psi = U - V,$$

and set $E = \{z \in D: U > V\}$ (E open). Since under the transformation (3) $\partial D \rightarrow \partial D_r$, from Lemma 1 $K_{D_r}(z, \bar{z})$ becomes infinite on ∂D_r . Thus since U is continuous on $\bar{D}_r \subset D$ and V on D_r , $\bar{E} \subset D_r$. Let O be any component of E . Then $\bar{O} \subset \bar{E} \subset D_r$ so that \bar{O} is compact. Thus the continuous function Ψ takes its maximum at a point $z_0 \in \bar{O}$ but $\Psi(z_0) = \max \Psi > 0$ so that $z_0 \in E$. Since Ψ has a maximum on E ,

$$\frac{\partial^2 \Psi}{\partial z^\alpha \partial \bar{z}^\beta} u^\alpha \bar{u}^\beta \leq 0$$

at z_0 for any vector (u^α) or by (15)

$$(16) \quad \frac{\partial^2 U}{\partial z^\alpha \partial \bar{z}^\beta} u^\alpha \bar{u}^\beta \leq \frac{\partial^2 V}{\partial z^\alpha \partial \bar{z}^\beta} u^\alpha \bar{u}^\beta.$$

From (15), (13) and (9a) and (c)

$$(17) \quad \frac{\partial^2 U}{\partial z^\alpha \partial \bar{z}^\beta} u^\alpha \bar{u}^\beta = -r_{\mu\bar{\nu}} \tilde{u}^\mu \bar{\tilde{u}}^\nu \geq 0,$$

where $(\tilde{u}^\mu) = ((\partial w^\mu / \partial z^\alpha) u^\alpha)$. From the definition of V the matrix $A = (\partial^2 V / \partial z^\alpha \partial \bar{z}^\beta)$ is positive definite. Hence by a classical theorem on the simultaneous reduction of a pair of hermitian quadratic forms there exists a nonsingular matrix T such that

$$A = T\bar{T}', B = T\Lambda\bar{T}', \Lambda = [\lambda_1, \dots, \lambda_n]$$

[7, 191], $B = (\partial^2 U / \partial z^\alpha \partial \bar{z}^\beta)$, and from (16) and (17)

$$0 \leq \lambda_i \zeta^i \bar{\zeta}^i \leq \sum_{i=1}^n \zeta^i \bar{\zeta}^i$$

where $\zeta = uT$. By taking ζ successively equal to $(1, 0, \dots, 0)$, $(0, 1, 0, \dots, 0)$, \dots , $(0, \dots, 0, 1)$, we get $0 \leq \lambda_i \leq 1$ ($i = 1, \dots, n$). Hence at z_0 since $\det T\bar{T}' > 0$

$$0 \leq \det \left(\frac{\partial^2 U}{\partial z^\alpha \partial \bar{z}^\beta} \right) \leq \det \left(\frac{\partial^2 V}{\partial z^\alpha \partial \bar{z}^\beta} \right).$$

But from (15) and (14)

$$\det \left(\frac{\partial^2 U}{\partial z^\alpha \partial \bar{z}^\beta} \right) \geq G_D(z, \bar{z})$$

while from (15), (2), (11) and (12)

$$\det \left(\frac{\partial^2 V}{\partial z^\alpha \partial \bar{z}^\beta} \right) = T_{D_r}(z, \bar{z}) = K_{D_r}(z, \bar{z}) \frac{T_D(0, 0)}{K_D(0, 0)}$$

so that at z_0

$$(18) \quad \frac{G_D(z, \bar{z})}{T_D(0, 0)} \leq \frac{K_{D_r}(z, \bar{z})}{K_D(0, 0)},$$

that is, $U \leq V$, which is a contradiction. Thus O and E are empty so that $U \leq V$ on D_r , and relation (18) holds on D_r . Since K_{D_r} is a relative invariant,

$$\frac{G_D(z, \bar{z})}{T_D(0, 0)} \leq \frac{r^{-2n} K_D \left(\frac{z}{r}, \frac{\bar{z}}{r} \right)}{K_D(0, 0)}$$

and letting $r \rightarrow 1$ from the continuity of $K_D(z, \bar{z})$ on D follows

$$\frac{G_D(z, \bar{z})}{T_D(0, 0)} \leq \frac{K_D(z, \bar{z})}{K_D(0, 0)}$$

which gives (10) by (13), (4) and the fact that $I_D(0, 0) = I_D(z, \bar{z})$.

To prove the last part of the theorem we note that $d\sigma_\Delta = ds_\Delta$ and $g_\Delta = T_\Delta$ and since T_Δ is a relative invariant equality follows in (10).

REMARK. The inequality of Theorem 1 is invariant under biholomorphic mappings in the following sense. Let $D \in \mathcal{D} \rightarrow D^*$ and $\Delta \in \mathcal{K} \rightarrow \Delta^*$ under the biholomorphic maps $z^* = z^*(z)$ and $w^* = w^*(w)$ respectively. Suppose there exists a biholomorphic mapping $w^* = f(z^*)$ of D^* into Δ^* . Then

$$(19) \quad g_{\Delta^*}(w^*, \bar{w}^*) |J_f(z^*)|^2 \leq T_{D^*}(z^*, \bar{z}^*)$$

for $z^* \in D^*$.

Proof. Set $h = w^{*-1}fz^*$. Then $w = h(z)$ is a biholomorphic mapping of D into Δ and by Theorem 1

$$g_\Delta(w, \bar{w}) |J_h(z)|^2 \leq T_D(z, \bar{z}).$$

Since

$$T_D(z, \bar{z}) = T_{D^*}(z^*, \bar{z}^*) |J_{z^*}(z)|^2 \text{ and } g_\Delta(w, \bar{w}) = g_{\Delta^*}(w^*, \bar{w}^*) |J_{w^*}(w)|^2,$$

and from the definition of h

$$|J_h(z)|^2 = |J_f(z^*)|^2 |J_{z^*}(z)|^2 |J_{w^*}(w)|^{-2},$$

from which (19) follows.

The following form of Theorem 1 is not invariant under biholomorphic mappings. It may be proved by replacing U in (15) by $U = \log G_D(z, \bar{z})$.

THEOREM 2. *If there exists a biholomorphic mapping $w = w(z)$ of a domain $D \in \mathcal{D}$ into a Kähler manifold $\Delta \in \mathcal{K}$ with condition (9b) replaced by*

$$\det(-r_{\alpha\bar{\beta}}) \geq T_D(0, 0)g_\Delta(w, \bar{w}),$$

then on D

$$g_\Delta(w, \bar{w}) |J_w(z)|^2 \leq T_D(z, \bar{z})/T_D(0, 0).$$

REMARKS.

1. The result obtained by Dinghas in [4] is a special case of Theorem 2 when D is the unit hypersphere.

2. A disadvantage of Theorem 2 is the fact that the original domain D furnished with the Bergman metric cannot belong to our admissible class of Kähler manifolds Δ .

3. Further extensions and applications of Theorem 1.

1. *Area theorem for manifolds of n real dimensions.* We also may derive an inequality for manifolds of n (real) dimensions. Let $M \subset D$ be a continuously differentiable manifold of dimension n parametrized by $z^i = z^i(u)$, u a point of the n -cube I^n . The n -dimensional noneuclidean analytic volume of M is

$$dV_M(z) = T_D^{1/2}(z, \bar{z}) |\partial(z^1, \dots, z^n)/\partial(u^1, \dots, u^n)| d\omega_u,$$

and is invariant under biholomorphic mappings of D [5, 330]. Then

THEOREM 3. *Let M be a continuously differentiable manifold of real dimension n in D and $w(M)$ its image under a biholomorphic mapping of $D \in \mathcal{D}$ into $\Delta \in \mathcal{K}$. Then $V_\Delta(w(M)) \leq V_D(M)$ and equality holds if the mapping is onto and $d\sigma_\Delta = ds_D$.*

We note that if we take $\Delta = D$ and $d\sigma_\Delta = ds_D$, we obtain a generalization of the Schwarz-Pick lemma to this real n -dimensional noneuclidean volume (see also Theorem 25.1 in [5]).

Theorem 3 is invariant under biholomorphic mappings of D and Δ in the sense described in §2.2 since all quantities involved are invariant under such biholomorphic mappings.

2. *Properties of certain domain functions.* The following theorems give some useful inequalities connecting the relative invariants of a domain D and $I_D(z, \bar{z})$. The first theorem follows easily from Theorem 1 for homogeneous star-shaped domains but also holds for any bounded domain.

THEOREM 4. *Let D be any bounded domain and $w = w(z)$ a biholomorphic mapping of D into D . The invariant $I_D(z, \bar{z}) \leq 1$ on D if and only if $K_D(w, \bar{w}) |J_w(z)|^2 \leq T_D(z, \bar{z})$.*

Proof. Suppose that $I_D(z, \bar{z}) \leq 1$ on D and assume that the conclusion does not hold, that is, there is a point $z_0 \in D$ such that

$$K_D(w_0, \bar{w}_0) |J_w(z_0)|^2 > T_D(z_0, \bar{z}_0) \quad (w_0 = w(z_0)).$$

Now $D^* = w(D) \subset D$ and hence $K_{D^*}(w_0, \bar{w}_0) \geq K_D(w_0, \bar{w}_0)$ [2, 45]. Therefore

$$K_{D^*}(w_0, \bar{w}_0) |J_w(z_0)|^2 = K_D(z_0, \bar{z}_0) > T_D(z_0, \bar{z}_0) \quad \text{or} \quad I_D(z_0, \bar{z}_0) > 1$$

which is a contradiction. Since the Jacobian of the identity mapping is 1, the converse of the theorem is trivial.

A useful application of Theorem 4 is

THEOREM 5. *Let D be a bounded complete circular domain in C^n with center at the origin and $w = w(z)$ a biholomorphic mapping of D into D . If $I_D \leq 1$ on D , then $|J_w(z)|^2 \leq \omega(D) T_D(z, \bar{z})$. Also $\omega(w(G)) \leq \omega(D) V_D(G)$ (ω euclidean volume) for any measurable $G \subset D$, where $V_D(G) = \int_G T_D(z, \bar{z}) d\omega_z$.*

Proof. Since D is a bounded complete circular domain, $K_D(z, \bar{z})$ attains its minimum at $z=0$ [6, 79] and the minimum value is $1/\omega(D)$. Then the conclusions of the theorem follow from Theorem 4.

Since $V_D(G) = V_D(w(G))$ we have

COROLLARY. *Under the hypotheses of Theorem 5*

$$\omega(w(G))/V_D(w(G)) \leq \omega(D)$$

for any measurable set $G \subset D$ with nonzero measure. In particular if $w = w(z)$ is the identity mapping, then

$$\omega(G)/V_D(G) \leq \omega(D).$$

We remark that $I_D < 1$ for the classical Cartan domains. (See §4.) In fact we do not know examples of domains for which $I_D(z_0, \bar{z}_0) \geq 1$ at some point $z_0 \in D$.

Finally if we apply Theorem 1 under the identity mapping $w = z$, we get

THEOREM 6. *Let Δ be a homogeneous bounded domain in C^n and D a subdomain which is equivalent to a domain in \mathcal{D} . Then for $z \in D$*

$$T_\Delta(z, \bar{z}) \leq T_D(z, \bar{z}).$$

4. Relative invariants on the classical Cartan domains.

1. The theorems in §2 give interesting results for the classical Cartan domains. These domains along with two special domains are the 4 types of bounded irreducible symmetric domains in $C^{n'}$, into which all bounded symmetric domains in $C^{n'}$ can be mapped biholomorphically. Let z be a matrix of complex elements, z' its transpose, z^* its conjugate transpose, and I the identity matrix. The first 3 types are represented by

$$R_j = [z: I - zz^* > 0]$$

($j=I, II, III$) where z is a matrix of type (n, m) on R_I , z is a symmetric matrix of order n on R_{II} and a skew-symmetric matrix of order n on R_{III} , and “ > 0 ” means that the quadratic form is positive definite. The fourth type R_{IV} is the set of n dimensional vectors such that

$$|zz'| < 1, 1 - 2\bar{z}z' + |zz'|^2 > 0.$$

These domains belong to class \mathcal{D} . The Bergman kernel function of these domains is known [6] so that to get inequality (2.10) it is sufficient to find $T_j = T_{R_j}(0, 0)$ and use formula (2.11). For the first 3 types

$$K_j(z, z^*) = \frac{1}{\omega_j \det^p(I - zz^*)},$$

$p = m + n$ for R_I , $n + 1$ for R_{II} and $n - 1$ for R_{III} and ω_j is the euclidean volume of R_j . For R_{IV}

$$K_{IV}(z, \bar{z}) = \frac{1}{\omega_{IV}(1 + |zz'|^2 - 2\bar{z}z')^n}.$$

In case I since $\log K_I(z, z^*) = -(m + n) \log Q_I - \log \omega_I$, where $Q_I = \det(I - zz^*)$ we need the value of $\partial^2 Q_I / \partial z^{\alpha\beta} \partial \bar{z}^{\gamma\delta}$ at $z = 0$. To evaluate this use the expansion for the characteristic equation of zz^* with $\lambda = 1$:

$$\det(\lambda I - zz^*) = \lambda^n - \sigma \lambda^{n-1} + t_{n-2} \lambda^{n-2} - \dots \pm \det zz^*,$$

where t_i is the sum of the principal i -rowed minors of zz^* , σ being the trace. Since t_i is a homogeneous polynomial of degree $2i$, only the second derivatives of σ

contribute nonzero terms at $z=0$. Also all the first derivatives of t_i and σ are 0 at $z=0$. Now

$$\frac{\partial}{\partial z^{\alpha\beta}} \log K_I(z, z^*) = -(m+n)Q_I^{-1} \frac{\partial}{\partial z^{\alpha\beta}} Q_I,$$

$$\frac{\partial^2}{\partial z^{\alpha\beta} \partial \bar{z}^{\gamma\delta}} \log K_I(z, z^*) = (m+n)Q_I^{-2} \frac{\partial}{\partial z^{\alpha\beta}} Q_I \frac{\partial}{\partial \bar{z}^{\gamma\delta}} Q_I - (m+n)Q_I^{-1} \frac{\partial^2 Q_I}{\partial z^{\alpha\beta} \partial \bar{z}^{\gamma\delta}}.$$

These remarks and formulas apply also to cases II and III.

Since $\sigma = \sum_{j,k} z^{jk} \bar{z}^{jk}$, at $z=0$

$$\frac{\partial^2 \sigma}{\partial z^{\alpha\beta} \partial \bar{z}^{\gamma\delta}} = \delta_{j\alpha} \delta_{k\beta} \delta_{j\gamma} \delta_{k\delta},$$

$$\frac{\partial^2 \log K_I(z, z^*)}{\partial z^{\alpha\beta} \partial \bar{z}^{\alpha\beta}} = m+n,$$

and all other derivatives are zero so that

$$T_I(0, 0) = \det[(m+n)I] = (m+n)^{mn},$$

$$I_I = \frac{1}{\omega_I(m+n)^{mn}},$$

and from (2.4)

$$(1) \quad T_I(z, z^*) = \frac{(m+n)^{mn}}{\det^{m+n}(I - zz^*)}.$$

The trace of zz^* for a symmetric matrix is

$$\sigma = \sum_j z^{jj} \bar{z}^{jj} + 2 \sum_{j \neq k} z^{jk} \bar{z}^{jk}$$

so that at $z=0$

$$\frac{\partial^2 \sigma}{\partial z^{jj} \partial \bar{z}^{jj}} = 1, \quad \frac{\partial^2 \sigma}{\partial z^{jk} \partial \bar{z}^{jk}} = 2 \quad (j \neq k)$$

and all other derivatives of σ are zero. Thus

$$T_{II}(0, 0) = (n+1)^{n((n+1)/2)} 2^{n(n-1)/2},$$

$$I_{II} = \frac{1}{\omega_{II} 2^{n(n-1)/2} (n+1)^{n(n+1)/2}}$$

and

$$(2) \quad T_{II}(z, z^*) = \frac{2^{n(n-1)/2} (n+1)^{n(n+1)/2}}{\det^{n+1}(I - z\bar{z})}.$$

For a skew symmetric matrix $\sigma = 2 \sum_{j < k} z^{jk} \bar{z}^{jk}$, where the matrix z has only $n(n-1)/2$ distinct nonzero elements, so that we get

$$(3) \quad \begin{aligned} T_{\text{III}}(0, 0) &= [2(n-1)]^{n(n-1)/2}, & I_{\text{III}} &= \frac{1}{\omega_{\text{III}}[2(n-1)]^{n(n-1)/2}}, \\ T_{\text{III}}(z, \bar{z}) &= \frac{[2(n-1)]^{n(n-1)/2}}{\det^{n-1}(I + z\bar{z})}. \end{aligned}$$

For case IV setting $\Delta = 1 + |zz'|^2 - 2\bar{z}z'$

$$\frac{\partial}{\partial z^\alpha} \Delta = 2z^\alpha [(\bar{z}^1)^2 + \dots + (\bar{z}^n)^2] - 2z^\alpha,$$

which is 0 at $z=0$ and similarly for $\partial\Delta/\partial\bar{z}^\alpha$ and

$$\frac{\partial^2 \Delta}{\partial z^\alpha \partial \bar{z}^\beta} = 4z^\alpha \bar{z}^\beta - 2\delta_{\alpha\beta}$$

so that at $z=0$

$$\frac{\partial^2 \log K_{\text{IV}}(z, \bar{z})}{\partial z^\alpha \partial \bar{z}^\beta} = -n(-2\delta_{\alpha\beta})$$

and

$$(4) \quad \begin{aligned} T_{\text{IV}}(0, 0) &= (2n)^n, & I_{\text{IV}} &= \frac{1}{\omega_{\text{IV}}(2n)^n}, \\ T_{\text{IV}}(z, \bar{z}) &= \frac{(2n)^n}{(1 + |zz'|^2 - 2\bar{z}z')^n}. \end{aligned}$$

Formulas (1), (2) and (3) give the interesting result

THEOREM 7. *The function $\log \det^{-1}(I - zz^*)$ satisfies the partial differential equation*

$$(5) \quad \det\left(\frac{\partial^2 V}{\partial z^\alpha \partial \bar{z}^\beta}\right) = ae^{bv}$$

on R_j ($j=I, II, III$) where $a=1$, $b=m+n$ for an (n, m) matrix, $a=2^{n(n-1)/2}$ for a symmetric or skew-symmetric matrix and $b=n+1$ for a symmetric and $n-1$ for a skew-symmetric matrix. For case IV $-\log(1 + |zz'|^2 - 2\bar{z}z')$ satisfies the partial differential equation

$$(6) \quad \det\left(\frac{\partial^2 V}{\partial z^\alpha \partial \bar{z}^\beta}\right) = 2^n e^{nv}$$

on R_{IV} .

This theorem corresponds to Lemma 4 of Dinghas [4] for the function

$$-\log(1 - z^j \bar{z}^j).$$

Using the values for the euclidean volume given in [6] and induction on n we find that the invariant $I_j < 1$ ($j=I, II, III, IV$). Thus Theorem 5 holds for the

Cartan domains and from this result, Theorem 4.2.1 of [6] and the expressions (1)–(4) for $T_j(z, z^*)$ we obtain an interesting distortion theorem on the Jacobian of an interior mapping of a Cartan domain:

THEOREM 8. *Let $w = w_j(z)$ be a biholomorphic mapping of the Cartan domain R_j into itself. Then*

$$|J_{w_j}(z)|^2 \leq \frac{|J_j(z, \bar{z})|^2}{I_j}$$

for $z \in R_j$, where I_j is the invariant of R_j and $J_j(z, \bar{z})$ the Jacobian of the holomorphic automorphism of R_j which maps z into the origin ($j = \text{I, II, III, IV}$).

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